L30 March 31 Htpy Equiv

Monday, March 23, 2015 5:29 PM

We saw that $\mathbb{R}^n$ is special that

$[X, \mathbb{R}^n]$ is a singleton $\forall$ space X.

Let Y be such a space, i.e.

$[X, Y]$ is a singleton $\forall$ space X

$\Downarrow$

$[Y, Y]$ is a singleton

$\Downarrow$

The maps $id \simeq c : Y \longrightarrow Y$ where

$id(y) = y$ and $c(y) = y_0$ $\forall y \in Y$

$\Downarrow$

$[X, Y]$ is a singleton $\forall$ space X

Let $f : X \longrightarrow Y$

Then $f = id \circ f \simeq c \circ f = c$

Definition.

A space Y is contractible if the identity map $id \simeq$ constant map.

Example.

- $\mathbb{R}^n$ is contractible
- Any star-shape $X \subset \mathbb{R}^n$ is contractible

From above

X is contractible $\Leftrightarrow$ $[W, X] = \{[c]\}$ $\forall W$

L30 p2

Tuesday, March 24, 2015 10:57 AM

Given a contractible space X, i.e., $id: X \longrightarrow X$ is homotopic to c onto $x_0 \in X$

Qu. What can you say about X and $\{x_0\}$?

- $X \neq \{x_0\}$, not homeomorphic
- Somehow, they are very similar.

Observe that $$\begin{tikzcd} \{x_0\} \arrow[r, shift left, "i"] & X \arrow[l, shift left, "c"] \end{tikzcd}$$, we have

$$\underbrace{c \circ i = id_{\{x_0\}}}_{\text{trivial by def.}} \quad \text{and} \quad i \circ c = \underbrace{c \simeq id}_{X \text{ is contractible}}{}_X$$

Though i and c are not bijections, they play a very similar role as inverses.

Homotopy Equivalences.

Let X, Y be spaces, and $f: X \longrightarrow Y$, $g: Y \longrightarrow X$ satisfy $g \circ f \simeq id_X$ and $f \circ g \simeq id_Y$. We call f, g homotopy equivalences (homotopy inverse to each)

The space X, Y are homotopy equivalent or they are of the same homotopy type.

Notation $X \simeq Y$.

f and g are called homotopy inverses to each other.

L30 p3

Monday, March 30, 2015 4:43 PM

Fact. $X \overset{\text{homeo}}{=\!=} Y \Longrightarrow X \simeq Y$

This is obvious because $f \circ f^{-1} = id \simeq id$

Theorem

(i) $X_1 \simeq X_2 \Rightarrow [X_1, Y] \xleftrightarrow{\text{bijection}} [X_2, Y]$

(ii) $Y_1 \simeq Y_2 \Rightarrow [X, Y_1] \longleftrightarrow [X, Y_2]$

Key step It again comes from

$$\begin{tikzcd} X \arrow[r, bend left, "f_0"] \arrow[r, bend right, "f_1"'] & Y \arrow[r, bend left, "g_0"] \arrow[r, bend right, "g_1"'] & Z \end{tikzcd}$$

$f_0 \simeq f_1$ and $g_0 \simeq g_1 \Longrightarrow g_0 f_0 \simeq g_1 f_1$ etc.

(ii) If we have $\begin{tikzcd} Y_1 \arrow[r, bend left, "\varphi"] & Y_2 \arrow[l, bend left, "\psi"] \end{tikzcd}$ then

$$\begin{tikzcd} {[X, Y_1]} \arrow[r, "\varphi_\#"] & {[X, Y_2]} \\ {[f]} \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & {[\varphi \circ f]} \arrow[u, phantom, sloped, "\in"] \\ {[X, Y_1]} & {[X, Y_2]} \arrow[l, "\psi_\#"'] \\ {[\psi \circ g]} \arrow[u, phantom, sloped, "\in"] & {[g]} \arrow[u, phantom, sloped, "\in"] \arrow[l, mapsto] \end{tikzcd}$$

both are defined

Moreover,

$\psi_\# \circ \varphi_\# = id$ because $\psi \circ \varphi \circ f \simeq (id)_{Y_1} \circ f = f$

$\varphi_\# \circ \psi_\# = id$ because $\varphi \circ \psi \circ g \simeq (id)_{Y_2} \circ g = g$

(i) For $X_1 \xrightarrow{\varphi} X_2$, it is similar, but the mapping reverses, $[X_1, Y] \xleftarrow{\varphi^*} [X_2, Y]$

L30 p4

Tuesday, March 24, 2015 11:21 AM

Examples.

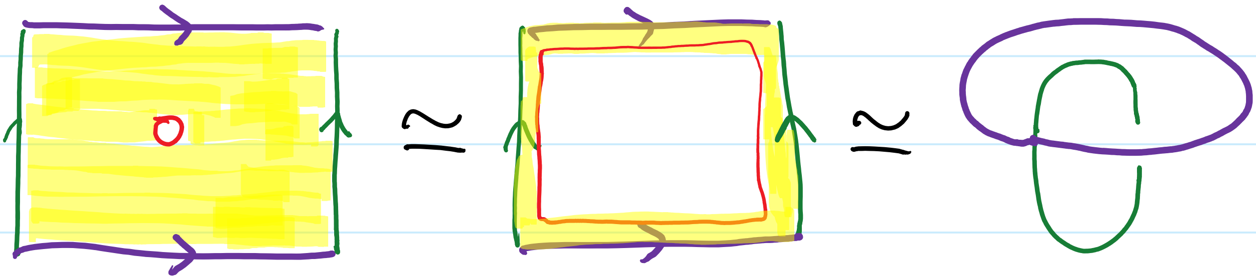
(1) $\mathbb{R}^n \simeq \{0\}$

(2) $S^1 \times \mathbb{R} \simeq S^1$

$$(e^{i\theta}, t) \xmapsto{f} e^{i\theta} \xmapsto{g} (e^{i\theta}, 0) \xmapsto{f} e^{i\theta}$$

$$fg = id_{S^1}, \quad gf \simeq id_{S^1 \times \mathbb{R}}$$

(3) Punctured Torus $\simeq S^1 \vee S^1 = \infty$



Definition $A \subset X$ is called a retract if $\exists$ continuous $r : X \longrightarrow A$ such that $r|_A = id_A$

In other words, $A \overset{i}{\underset{r}{\rightleftarrows}} X$ satisfies $r \circ i = id_A$

The other composition $i \circ r$ has no additional fact.

If, in addition, $i \circ r \simeq id_X$, then A is a deformation retract. In this case, $A \simeq X$.

The above examples

$$\mathbb{R}^n \simeq \{0\}, \quad S^1 \times \mathbb{R} \simeq S^1, \quad \mathbb{T}^2 \setminus \{*\} \simeq S^1 \vee S^1$$

are deformation retracts

L30 p5

Monday, March 30, 2015 9:45 PM

Example

Let $X = S^2 \setminus \{p, q\}$ where $p_3, q_3 > 0$

and $A = \{(x_1, x_2, x_3) \in S^2 : x_3 \leq 0\}$

Then $A \subset X$ is a retract by

$$r : X \longrightarrow A : (x_1, x_2, x_3) \longmapsto (x_1, x_2, -x_3)$$

Intuitively, one sees that $X \not\simeq A$

History By definition, it is clear that

$$\begin{tikzcd} X \overset{\text{homeo}}{=} Y \arrow[r, Rightarrow] & X \simeq Y \arrow[l, bend left, "\text{Qu}"] \end{tikzcd}$$

additional condition?

Poincaré Conjecture If $X \simeq S^n$ then $X = S^n$

$n = 0, 1, 2$	Relatively Easy
$n = 3$	Original Question of Poincaré
$n \geq 5$	Stephen Smale, 1966 Fields Medal
$n = 4$	Michael Freedman, 1982 Fields Medal
$n = 3$	Grigori Perelman, 2003 announced

Base point

Recall that $[S^0, X]$ is in principle counting the number of path components of X. But it counts better for $(S^0, 1) \longrightarrow (X, x_0)$

i.e. $f : \{\pm 1\} \longrightarrow X$ with $f(1) = x_0$

The role of x_0 is a base point

Let X be a space with a base point $x_0 \in X$.
A loop at x_0 is a continuous path

$$\gamma : [0,1] \longrightarrow X \quad \text{with} \quad \gamma(0) = \gamma(1) = x_0.$$

In other words, we are considering

$$([0,1], \{0,1\}) \xrightarrow{\gamma} (X, \{x_0\})$$

Another alternative is to study

$$(S^1, 1) \xrightarrow{\gamma} (X, x_0)$$

In any formulation, the aim is to fixed the mappings (curves) at the base point x_0. This is also needed in the homotopy

Given two loops $\gamma_0, \gamma_1 : [0,1] \longrightarrow X$ at x_0, they are loop homotopic if there exists a loop homotopy $L : [0,1] \times [0,1] \longrightarrow X$ such that

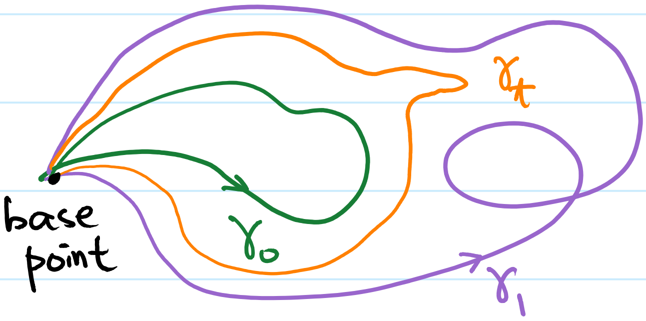
loop parameter homotopy time

$$\left.\begin{array}{l} L(s,0) = \gamma_0(s) \\ L(s,1) = \gamma_1(s) \end{array}\right\} \quad s \in [0,1]$$

normal homotopy

$$L(0,t) = L(1,t) = x_0 \quad \forall \; t \in [0,1]$$

at any time t, it is a loop based at x_0



For $f, g : (X, A) \longrightarrow (Y, B)$ with $f|_A = g|_A$

A homotopy rel A is a continuous mapping

$$H : X \times [0,1] \longrightarrow Y$$

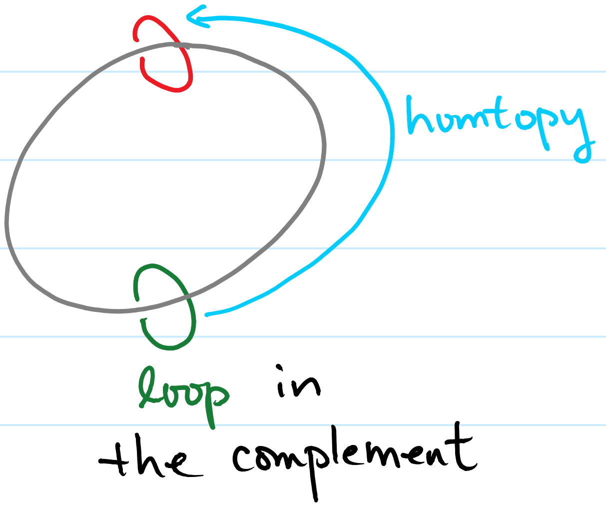
$$H(x,0) = f(x), \quad H(x,1) = g(x), \quad x \in X$$

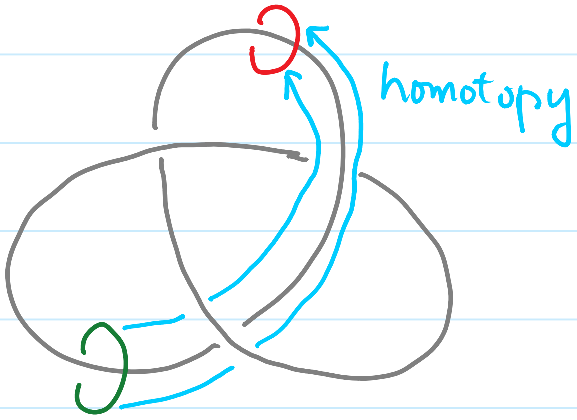
$$H(x,t) = f(x) = g(x), \quad x \in A, \quad t \in [0,1]$$

Example of importance of base point

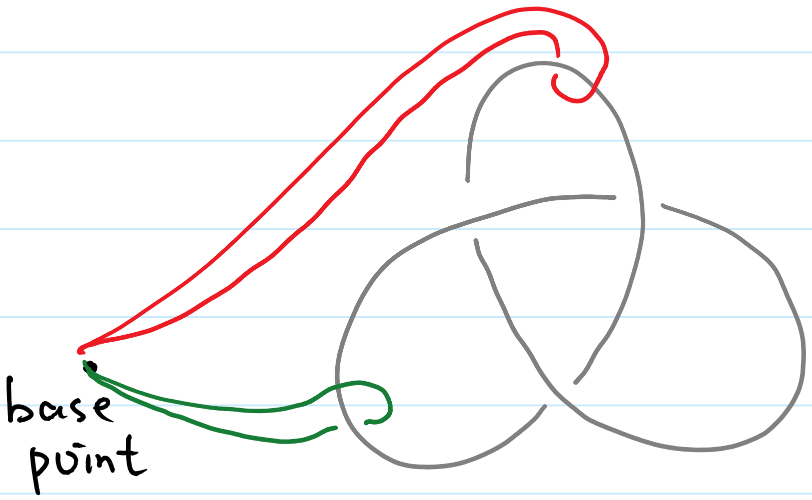
In topology, we often need to study complement of a knot

$\mathbb{R}^3 \setminus$ circle or $\mathbb{R}^3 \setminus$ Trefoil





If the loops have no base point, both $\mathbb{R}^3 \setminus$ circle, $\mathbb{R}^3 \setminus$ Trefoil has one homotopy class



These two loops with base point are not homotopic. Therefore

$$\mathbb{R}^3 \setminus \text{circle} \neq \mathbb{R}^3 \setminus \text{Trefoil}$$